ACTION OF THE SYMPLECTIC GROUP ON THE SIEGEL UPPER HALF-PLANE

1. Introduction

In this note, we study the action of the symplectic group on the Siegel upper half-plane. This action can be viewed as a generalisation of the action of $\mathsf{SL}(2,\mathbb{R})$ on the complex upper-half plane by Möbius transformations. Our treatment is based on [2] (another good reference is [1]). Let us start by recalling the $\mathsf{SL}(2,\mathbb{R})$ action.

Given a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}(2,\mathbb{R})$ and a point $z \in \mathfrak{h} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$, we define the action as a Möbius transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.$$

This transformation maps upper half-plane to itself, thanks to the formula

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2}.$$

The action is transitive. For example, we can carry i to any $z = x + iy \in \mathfrak{h}$ by

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}.$$

2. Definitions and results

Let us now define the symplectic group.

Definition 2.1. The *Symplectic Group* is the group of all matrices $M \in \mathsf{GL}(2n,\mathbb{R})$ satisfying $M^\top J M = J$; with $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I_n is the $n \times n$ identity matrix. We denote it by $\mathsf{Sp}(2n,\mathbb{R})$.

Decomposing M into four $n \times n$ blocks $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we see that $M^\top J M = J$ is equivalent to the conditions

$$\boxed{A^\top C \text{ and } B^\top D \text{ are symmetric and } A^\top D - C^\top B = I_n.} \tag{1}$$

For $n = 1$, $\mathsf{Sp}(2,\mathbb{R})$ is the same as $\mathsf{SL}(2,\mathbb{R})$. We will now prove this.

Proposition 2.2. $\mathsf{Sp}(2,\mathbb{R}) = \mathsf{SL}(2,\mathbb{R})$.

Proof. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{GL}(2,\mathbb{R})$. Notice that

$$M^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & ad-bc \\ bc-ad & 0 \end{pmatrix}.$$

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So, $M^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ if and only if $\det M = ad - bc = 1$. In other words, $\mathsf{Sp}(2,\mathbb{R}) = \mathsf{SL}(2,\mathbb{R})$. □

Recall that a *symmetric matrix* is a square matrix that is equal to its transpose. Let $\mathrm{Sym}_n(\mathbb{F})$ denote the set of all $n \times n$ symmetric matrices with entries in an arbitrary field $\mathbb{F}$.

A real symmetric matrix A is called *positive definite* if $v^\top A v > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$. We write $A > 0$ if A is positive definite. It is well-known that if A is positive definite, then A has a unique positive definite square root, that is, there is one and only one positive definite matrix B such that $B^2 = A$. We denote it by $\sqrt{A}$.

Definition 2.3. The *Siegel upper half-plane* is the set of all complex symmetric $n \times n$ matrices with positive definite imaginary part. We denote it by $\mathbf{SH}_n$:

$$\mathbf{SH}_n := \{X + iY \in \mathrm{Sym}_n(\mathbb{C}) \mid X, Y \in \mathrm{Sym}_n(\mathbb{R}), Y > 0\}.$$

Let $\mathsf{Sp}(2n,\mathbb{R})$ acts on $\mathbf{SH}_n$ by the rule:

$$M \cdot Z := (AZ + B)(CZ + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathsf{Sp}(2n,\mathbb{R})$ and $Z = X + iY \in \mathbf{SH}_n$.

Let $P = AZ + B$ and $Q = CZ + D$. We need to verify that the matrix Q is invertible and the product indeed lies in $\mathbf{SH}_n$.

Let us pause for a moment and think about the $\mathsf{SL}(2,\mathbb{R})$ action. (We shall use lowercase letters when talking about the $\mathsf{SL}(2,\mathbb{R})$ action.) The element $q = cz + d$ is always invertible in that case, because the solution to $cz + d = 0$ lies in the real axis. However, this approach does not generalise to higher dimensions. A better method is to use the relation $2iy = p\bar{q} - q\bar{p}$. Since $y > 0$, it follows from the previous equality that $q \neq 0$.

Let us now compute $P^\top \overline{Q} - Q^\top \overline{P}$. Using (1), we get

$$P^\top \overline{Q} - Q^\top \overline{P} = Z - \overline{Z} = 2iY. \tag{2}$$

Let $v \in \mathbb{R}^n$ such that $Qv = 0$. Using (2), we obtain

$$v^\top (2iY) v = v^\top (P^\top \overline{Q} - Q^\top \overline{P}) v = 0.$$

Thus $v^\top Y v = 0$ and since Y is positive definite, we obtain $v = 0$. Therefore Q is invertible and $M \cdot Z$ is well-defined.

Now we claim the $M \cdot Z$ is in $\mathbf{SH}_n$. There are two things to verify here: (1) $M \cdot Z$ is complex symmetric; (2) its imaginary part is positive definite.

An easy computation using (1) shows that $P^\top Q = Q^\top P$. Then

$$P^\top = P^\top Q Q^{-1} = Q^\top P Q^{-1}. \tag{3}$$

and hence $(Q^{-1})^\top P^\top = PQ^{-1}$. Thus $M \cdot Z = PQ^{-1}$ is symmetric.

The imaginary part of PQ^{-1} is given by

$$\frac{1}{2i}(PQ^{-1} - \overline{PQ^{-1}}).$$

We have

$$\begin{aligned}\frac{1}{2i}(PQ^{-1}-\overline{PQ^{-1}}) &= \frac{1}{2i}(Q^{\top})^{-1}(Q^{\top}PQ^{-1}\overline{Q}-Q^{\top}\overline{P})\overline{Q^{-1}} \\ &= \frac{1}{2i}(Q^{-1})^{\top}(P^{\top}\overline{Q}-Q^{\top}\overline{P})\overline{Q^{-1}} && \text{(using (3))} \\ &= (Q^{-1})^{\top}Y\overline{Q^{-1}}. && \text{(from (2))}\end{aligned}$$

Let $\overline{Q^{-1}} = F$. Then we can write the imaginary part of PQ^{-1} as $\overline{F}^{\top}YF$. Write $F = F_1 + iF_2$, where F_1, F_2 are real matrices. Then

$$\begin{aligned}\overline{F}^{\top}YF &= (F_1^{\top} - iF_2^{\top})Y(F_1 + iF_2) \\ &= F_1^{\top}YF_1 + F_2^{\top}YF_2 + i(F_1^{\top}YF_2 - F_2^{\top}YF_1).\end{aligned}$$

Since $\overline{F}^{\top}YF$ is a real matrix, we obtain

$$\overline{F}^{\top}YF = F_1^{\top}YF_1 + F_2^{\top}YF_2.$$

From the above expression, it is clear that $\overline{F}^{\top}YF$ is positive definite, i.e., $v^{\top}\overline{F}^{\top}YFv > 0$ for all nonzero $v \in \mathbb{R}^n$. This shows that PQ^{-1} is in $\mathbf{SH}_n$.

Verifying that the action described above satisfies the group action axioms is left to the reader. Finally, let us prove that it is transitive. The proof is very similar to the $SL(2,\mathbb{R})$ case.

Let $Z = X + iY \in \mathbf{SH}_n$. Consider iI_n. We have

$$\begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y}^{-1} \end{pmatrix} iI_n = X + iY.$$

Using (1), we easily verify that $\begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix}$ and $\begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y}^{-1} \end{pmatrix}$ are in $Sp(2n,\mathbb{R})$. This proves the transitivity of the action.

References

[1] Freitas, P. J., On the Action of the Symplectic Group on the Siegel Upper Half Plane, PhD thesis, University of Illinois at Chicago, 1999.

[2] Klingen, H., Introductory Lectures on Siegel Modular Forms, Cambridge: Cambridge University Press, 1990.