FORM RINGS AND REGULAR SEQUENCES

WAYNE NG KWING KING AND SREEHARI SURESH BABU

1. Introduction

The purpose of this write-up is to describe the results in the paper, *Form rings and regular sequences* [8], by P. Valabrega and G. Valla. In it, they study the relationship between an ideal $\mathfrak{b} = (f_1, \ldots, f_r) \subset A$ and the form ideal $\mathfrak{b}^*$ of the associated graded ring $G_A(\mathfrak{a})$. The authors were motivated by a result of Hironaka on initial forms and they wanted to extend it to a general situation (see Remark 3.8). Hironaka's interest in these algebraic objects comes from their intimate connection with geometry. Hironaka, in his paper on the resolution of singularities over a field of characteristic 0, studied various properties of the associated graded ring to get a better understanding of singularities. We briefly describe this geometric connection now.

Let $X \subset \mathbf{A}^n$ be a variety defined by the ideal $J = (f_1, \ldots, f_r)$ and suppose that $0 \in X$. We define the tangent space to X at 0 as the variety defined by the homogeneous ideal generated by the linear forms of all $f \in J$. The tangent cone to X at 0 is a much finer invariant than the tangent space and is extremely useful when 0 is a singular point. For $f \in k[x_1, \ldots, x_n]$, let f^* denote the homogeneous component of f of the lowest degree (the leading form of f), and let J^* be the ideal generated by the leading forms of all $f \in J$. Then the tangent cone to X at 0 is the variety defined by the homogeneous ideal J^*. If R is the coordinate ring of X, then the coordinate ring of the tangent cone is the associated graded ring $G_{\mathfrak{m}}(R)$, where $\mathfrak{m}$ denotes the maximal ideal $(x_1, \ldots, x_n)$ of R (cf. Lemma 2.1).

More generally, let X be an abstract algebraic variety, x a point of X, and $(\mathscr{O}_{X,x}, \mathfrak{m})$ be the local ring of X at x. Then the tangent cone to X at x is the spectrum of the associated graded ring of $\mathscr{O}_{X,x}$ with respect to the maximal ideal: $G_{\mathfrak{m}}(\mathscr{O}) = \bigoplus_{n=0}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.

The main theorems in the paper are the following:

(i) A necessary and sufficient condition for the form ideal $\mathfrak{b}^*$ to be generated by the initial forms of the generators of $\mathfrak{b}$ (Theorem 2.3).
(ii) A necessary and sufficient condition for $\mathfrak{b}^*$ to be generated by a regular sequence (Theorem 3.3).

As applications of the above, we prove some results related to the Cohen-Macaulayness of $G_A(\mathfrak{a})$.

2. Form rings and ideals

Let $\mathfrak{a}$ be an ideal of a Noetherian ring A. The **form ring** of A relative to $\mathfrak{a}$, which is denoted by $G_A(\mathfrak{a})$, is defined to be the graded $A/\mathfrak{a}$-algebra

$$G_A(\mathfrak{a}) = \bigoplus_{n=0}^{\infty} \frac{\mathfrak{a}^n}{\mathfrak{a}^{n+1}} = \frac{A}{\mathfrak{a}} \oplus \frac{\mathfrak{a}}{\mathfrak{a}^2} \oplus \frac{\mathfrak{a}^2}{\mathfrak{a}^3} \oplus \cdots .$$

The multiplication in $G_A(\mathfrak{a})$ is defined as follows: if $a + \mathfrak{a}^{n+1} \in \mathfrak{a}^n/\mathfrak{a}^{n+1}$ and $b + \mathfrak{a}^{m+1} \in \mathfrak{a}^m/\mathfrak{a}^{m+1}$, then

$$(a + \mathfrak{a}^{n+1}) \cdot (b + \mathfrak{a}^{m+1}) := ab + \mathfrak{a}^{m+n+1} \in \mathfrak{a}^{m+n}/\mathfrak{a}^{m+n+1}.$$

This is easily seen to be independent of the choice of a and b.

Given $a \in A$, let $\nu(a)$ be the largest integer n such that $a \in \mathfrak{a}^n$. The **initial form** of a is defined to be the residue class of a in $\mathfrak{a}^{\nu(a)}/\mathfrak{a}^{\nu(a)+1}$ and is denoted by a^*. If $a \in \cap_{n \geqslant 1}\mathfrak{a}^n$, then we set $\nu(a) = \infty$ and $a^* = 0$. The map $a \mapsto a^*$ is not a homomorphism of abelian groups, but it behaves "almost" like a homomorphism. More precisely, if $a, b \in A$, then either $a^* + b^* = (a+b)^*$ or $a^* + b^* = 0$. Similarly, either $a^*b^* = (ab)^*$ or $a^*b^* = 0$.

Let $\mathfrak{b}$ be an ideal of A. The **form ideal** of $\mathfrak{b}$ relative to $\mathfrak{a}$ is defined to be the homogeneous ideal of $G_A(\mathfrak{a})$ generated by all the initial forms of the elements in $\mathfrak{b}$ and is denoted by $\mathfrak{b}^*$. The n-th graded component of $\mathfrak{b}^*$ is equal to $(\mathfrak{b} \cap \mathfrak{a}^n + \mathfrak{a}^{n+1})/\mathfrak{a}^{n+1}$.

Lemma 2.1. *With notations as above,*

$$G_A(\mathfrak{a})/\mathfrak{b}^* \cong G_{A/\mathfrak{b}}(\mathfrak{b} + \mathfrak{a}/\mathfrak{b}).$$

Proof. For us, a ring homomorphism between graded rings always means a degree-preserving map, so it is enough to prove the above isomorphism on the level of graded components. The latter follows from the following string of isomorphisms:

$$\frac{(\mathfrak{b} + \mathfrak{a}/\mathfrak{b})^n}{(\mathfrak{b} + \mathfrak{a}/\mathfrak{b})^{n+1}} \cong \frac{\mathfrak{b} + \mathfrak{a}^n}{\mathfrak{b} + \mathfrak{a}^{n+1}} = \frac{\mathfrak{b} + \mathfrak{a}^{n+1} + \mathfrak{a}^n}{\mathfrak{b} + \mathfrak{a}^{n+1}} \cong \frac{\mathfrak{a}^n}{(\mathfrak{b} + \mathfrak{a}^{n+1}) \cap \mathfrak{a}^n} = \frac{\mathfrak{a}^n}{\mathfrak{b} \cap \mathfrak{a}^n + \mathfrak{a}^{n+1}}.$$

□

Using the above isomorphism, it is easy to see that $\mathfrak{b}^* = G_A(\mathfrak{a})$ if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal. Thus, we will always assume that $\mathfrak{a}$ and $\mathfrak{b}$ are proper and are not comaximal.

If A is Noetherian, then $G_A(\mathfrak{a})$ is Noetherian. In fact, if $\mathfrak{a} = (a_1, \ldots, a_r)$ and $\alpha_i = a_i \bmod \mathfrak{a}^2$, then

$$G_A(\mathfrak{a}) \cong (A/\mathfrak{a})[\alpha_1, \ldots, \alpha_r].$$

In particular, $\mathfrak{b}^*$ is a finitely generated ideal of $G_A(\mathfrak{a})$. However, $\mathfrak{b}^*$ is generally *not* generated by the initial forms of a given set of generators of $\mathfrak{b}$.

Example 2.2. Let $A = k[\![X, Y, Z]\!]$, $\mathfrak{a} = (X, Y, Z)$ and $\mathfrak{b} = (XZ - Y^3, YZ - X^4, Z^2 - X^3Y^2)$. By abuse of notation, we write X, Y, Z to denote their own initial forms in $G_A(\mathfrak{a})$. Then $G_A(\mathfrak{a}) \cong k[X, Y, Z]$. Moreover, $(XZ - Y^3)^* = XZ$, $(YZ - X^4)^* = YZ$ and $(Z^2 - X^3Y^2)^* = Z^2$. However, XZ, YZ and Z^2 do not generate $\mathfrak{b}^*$. For example,

$$-Y(XZ - Y^3) + X(YZ - X^4) = Y^4 - X^5 \in \mathfrak{b},$$

so $(Y^4 - X^5)^* = Y^4 \in \mathfrak{b}^*$. In fact, $\mathfrak{b}^* = (XZ, YZ, Z^2, Y^4)$.

Let $\mathfrak{b} = (f_1, \ldots, f_r)$. Notice that the n-th homogeneous component of $(f_1^*, \ldots, f_r^*) \subset G_A(\mathfrak{a})$ is equal to $(\sum_{i=1}^r \mathfrak{a}^{n-p_i}f_i + \mathfrak{a}^{n+1})/\mathfrak{a}^{n+1}$. Thus, if $\mathfrak{a}^n \cap \mathfrak{b} = \sum_{i=1}^r \mathfrak{a}^{n-p_i}f_i$, then $\mathfrak{b}_n^* = (f_1^*, \ldots, f_r^*)_n$ for all $n \geqslant 0$, and hence $\mathfrak{b}^* = (f_1^*, \ldots, f_r^*)$. The following theorem says that this condition is necessary as well.

Theorem 2.3. *If $\mathfrak{a}$ and $\mathfrak{b} = (f_1, \ldots, f_r)$ are ideals of A, then $\mathfrak{b}^* = (f_1^*, \ldots, f_r^*)$ in $G_A(\mathfrak{a})$ if and only if for all $n \geqslant 0$ the following equality holds:*

$$\mathfrak{a}^n \cap \mathfrak{b} = \sum_{i=1}^r \mathfrak{a}^{n-p_i}f_i,$$

where $p_i = \nu(f_i)$, $i = 1, \dots, r$.

Proof. Suppose that $\mathfrak{b}^* = (f_1^*, \dots, f_r^*)$. Then $\mathfrak{b}_n^* = (f_1^*, \dots, f_r^*)_n$ for all $n \geqslant 0$, so we have

$$\mathfrak{a}^n \cap \mathfrak{b} \subseteq \sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i + \mathfrak{a}^{n+1} \text{ for all } n \geqslant 0.$$

Intersecting with $\mathfrak{b}$, we get

$$\mathfrak{a}^n \cap \mathfrak{b} \subseteq (\sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i + \mathfrak{a}^{n+1}) \cap \mathfrak{b} = \sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i + \mathfrak{a}^{n+1} \cap \mathfrak{b}.$$

Proceeding inductively, we see that

$$\mathfrak{a}^n \cap \mathfrak{b} \subseteq \bigcap_{t \geqslant 0} \sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i + \mathfrak{a}^{n+t} \cap \mathfrak{b}.$$

Now the Artin-Rees lemma guarantees the existence of an integer $q \geqslant 0$ such that $\mathfrak{a}^{n+t} \cap \mathfrak{b} = \mathfrak{a}^{n+t-q}(\mathfrak{a}^q \cap \mathfrak{b})$ for all $n + t \geqslant q$. Let $d \geqslant \max_{1 \leqslant i \leqslant r}\{n - p_i\}$. If $t \geqslant q - n + d$, then $n + t - q \geqslant d \geqslant n - p_i$ for each i, so

$$\mathfrak{a}^{n+t-q}(\mathfrak{a}^q \cap \mathfrak{b}) \subseteq \mathfrak{a}^d \mathfrak{b} \subseteq \sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i.$$

Thus,

$$\begin{aligned}\mathfrak{a}^n \cap \mathfrak{b} &\subseteq \bigcap_{t \geqslant q-n+d} \sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i + \mathfrak{a}^{n+t} \cap \mathfrak{b} \\ &= \bigcap_{t \geqslant q-n+d} \sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i + \mathfrak{a}^{n+t-q}(\mathfrak{a}^q \cap \mathfrak{b}) \subseteq \sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i.\end{aligned}$$

The other inclusion is trivial. □

Remark 2.4. If A is a local ring, the coarser relation

$$\mathfrak{a}^n \cap \mathfrak{b} \subseteq \bigcap_{t \geqslant 0} \sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i + \mathfrak{a}^{n+t}$$

and Krull's intersection theorem are enough to conclude.

Let us have a quick discussion about the height of the form ideal. We remark that if $\mathfrak{a} \subseteq \mathfrak{b}$, then $\operatorname{ht} \mathfrak{b} = \operatorname{ht} \mathfrak{b}^*$. Moreover, we have the following Krull's height theorem-type of result:

Proposition 2.5. *If* $\mathfrak{a}$ *and* $\mathfrak{b} = (f_1, \dots, f_r)$ *are ideals of* A *such that* $\mathfrak{a} + \mathfrak{b} \neq A$, *then* $\operatorname{ht}(\mathfrak{b}^*) \leqslant r$.

Proof. Let $\mathfrak{m}$ be a maximal ideal containing both $\mathfrak{a}$ and $\mathfrak{b}$. An application of Krull's height theorem and it's converse gives us the inequality $\operatorname{ht} \mathfrak{m} \leqslant \operatorname{ht}(\mathfrak{m}/\mathfrak{b}) + r$. Since $(\mathfrak{a}+\mathfrak{b})/\mathfrak{b} \subseteq \mathfrak{m}/\mathfrak{b}$, the height of the initial ideal $(\mathfrak{m}/\mathfrak{b})^* \subseteq G_{A/\mathfrak{b}}(\mathfrak{b} + \mathfrak{a}/\mathfrak{b})$ is the same as that of $\mathfrak{m}/\mathfrak{b}$ by the above remark. Under the natural isomorphism described in Lemma 2.1, $(\mathfrak{m}/\mathfrak{b})^*$ corresponds to the ideal $\mathfrak{m}^*/\mathfrak{b}^*$. Thus,

$$\operatorname{ht} \mathfrak{m} \leqslant \operatorname{ht}(\mathfrak{m}^*/\mathfrak{b}^*) + r \leqslant \operatorname{ht} \mathfrak{m}^* - \operatorname{ht} \mathfrak{b}^* + r = \operatorname{ht} \mathfrak{m} - \operatorname{ht} \mathfrak{b}^* + r,$$

and the desired inequality follows. □

Example 2.6. If in addition $\operatorname{ht}(f_1{}^*, \dots, f_r{}^*) = r$, then we see that $\operatorname{ht}\mathfrak{b}^* = r$. But this does not guarantee the equality $\mathfrak{b}^* = (f_1{}^*, \dots, f_r{}^*)$. For example, let $A = k[\![X, Y, Z]\!]/(XZ - Y^3, YZ - X^4, Z^2 - X^3Y^2) = k[\![x, y, z]\!]$, $\mathfrak{a} = (x, y, z)$ and $\mathfrak{b} = (x)$. Using Lemma 2.1 and Example 2.2, it is easy to see that $G_A(\mathfrak{a}) \cong k[T_1, T_2, T_3]/(T_1T_3, T_2T_3, T_3^2, T_2^4) = k[t_1, t_2, t_3]$. The initial form of x corresponds to the element t_1. We claim that $\operatorname{ht}(t_1) = 1$. Indeed, $\operatorname{ht}(t_1) \leqslant 1$ by Krull's principal ideal theorem. Next, notice that the unique minimal prime of $G_A(\mathfrak{a})$ is (t_2, t_3), and since $t_1 \notin (t_2, t_3)$, $\operatorname{ht}(t_1) = 1$. This in turn implies that $\operatorname{ht}(\mathfrak{b}^*) = 1$. However, $y^3 \in \mathfrak{a}^3 \cap \mathfrak{b}$, but $y^3 \notin \mathfrak{a}^2x$. Hence $\mathfrak{b}^* \neq (x^*)$ by Theorem 2.3.

3. Regular sequences in $G_A(\mathfrak{a})$

Let A be a Noetherian ring, $\mathfrak{a} \subseteq A$ an ideal and $f_1, \dots, f_r \in A$ such that $\mathfrak{a}$ and $(f_1, \dots, f_r)$ are not comaximal. Results on necessary and sufficient conditions for $f_1^*, \dots, f_r^*$ to be a $G_A(\mathfrak{a})$-sequence are looked into, especially in the case when A is local.

We fix the following notation: let $\mathfrak{b}_i := (f_1, \dots, f_i)$ for $i = 1, \dots, r$ and $\mathfrak{b}_0 := (0)$. Furthermore, let $\bar{I}$ be the topological closure of an ideal I with respect to the $\mathfrak{a}$-adic topology.

Proposition 3.1. *If $\mathfrak{a}$ and $\mathfrak{b} = (f_1, \dots, f_r)$ are ideals of A such that $f_1^*, \dots, f_r^*$ is a $G_A(\mathfrak{a})$-sequence, then $\mathfrak{b}^* = (f_1^*, \dots, f_r^*)$.*

Proof. We proceed by induction on r: when $r = 1$, $\mathfrak{b} = (f_1)$ and f_1^* is a nonzerodivisor. Therefore for every $g \in A$, we have $(gf_1)^* = g^*f_1^*$, which shows $\mathfrak{b}^* = (f_1^*)$. Suppose now that the statement is true for $r - 1$.

Let $a \in \mathfrak{a}^n \cap \mathfrak{b}$ and let t be the largest integer such that $a \in \mathfrak{b}_{r-1} + f_r\mathfrak{a}^t$ (if such a t does not exist, set $t = \infty$). Since $a \in \mathfrak{b}_{r-1} + f_r\mathfrak{a}^t$, one can write $a = x + f_ry$ for some $x \in \mathfrak{b}_{r-1}$ and some y with $\nu(y) = t$. If $t + p_r < n$, then $f_ry \in (\mathfrak{a}^n + \mathfrak{b}_{r-1}) \cap \mathfrak{a}^{t+p_r} \subseteq (\mathfrak{a}^{t+p_r+1} + \mathfrak{b}_{r-1}) \cap \mathfrak{a}^{t+p_r} = \mathfrak{a}^{t+p_r+1} + (\mathfrak{b}_{r-1} \cap \mathfrak{a}^{t+p_r})$. So $f_r^*y^* \in (\mathfrak{b}_{r-1})^{*}$[1], and since $(\mathfrak{b}_{r-1})^* = (f_1^*, \dots, f_{r-1}^*)$ by the induction hypothesis, this implies $y^* \in (\mathfrak{b}_{r-1})^*$ because by assumption f_r^* is a nonzerodivisor on $G_A(\mathfrak{a})/(f_1^*, \dots, f_{r-1}^*)$. Hence $y \in \mathfrak{b}_{r-1} \cap \mathfrak{a}^t + \mathfrak{a}^{t+1}$, which implies $a \in \mathfrak{b}_{r-1} + f_r\mathfrak{a}^{t+1}$, contradicting the maximality of t.

Therefore $t + p_r \geqslant n$ and $a \in (\mathfrak{b}_{r-1} + f_r\mathfrak{a}^t) \cap \mathfrak{a}^n \subseteq \mathfrak{b}_{r-1} \cap \mathfrak{a}^n + f_r\mathfrak{a}^{n-p_r} = (\sum_{i=1}^{r-1} \mathfrak{a}^{n-p_i}f_i) + f_r\mathfrak{a}^{n-p_r}$, where the last equality is obtained by the induction hypothesis and Theorem 2.3. □

Remark 3.2. The converse of Proposition 3.1 however is false, even when A is local. Consider the following example: let $A = k[\![X, Y]\!]/(XY) = k[\![x, y]\!]$, $\mathfrak{b} = (x)$, $\mathfrak{a} = (x, y)$; we have $\mathfrak{a}^n \cap \mathfrak{b} = (x^n, y^n) \cap (x) = (x^n) + (y^n) \cap (x) = (x^n) = \mathfrak{a}^{n-1}\mathfrak{b}$. Hence by Theorem 2.3, $\mathfrak{b}^* = (x^*)$. However $G_A(\mathfrak{a}) = k[T_1, T_2]/(T_1T_2) = k[t_1, t_2]$ and $x^* = t_1$ is a zerodivisor in $G_A(\mathfrak{a})$.

Theorem 3.3. *Let $\mathfrak{a}$ and $\mathfrak{b} = (f_1, \dots, f_r)$ be ideals of the Noetherian ring A. Then the following are equivalent:*

(i) *$(f_1^*, \dots, f_r^*)$ is a $G_A(\mathfrak{a})$-sequence;*
(ii) *For each $i = 1, \dots, r$, $(\mathfrak{b}_{i-1} : f_i) \subseteq \bar{\mathfrak{b}}_{i-1}$ and $\mathfrak{b}_i \cap \mathfrak{a}^n = \sum_{j=1}^{i} \mathfrak{a}^{n-p_j}f_j$ for all $n \geqslant 0$.*

Proof. (i) ⇒ (ii): By Proposition 3.1 and Theorem 2.3, it only remains to show that $(\mathfrak{b}_{i-1} : f_i) \subseteq \bar{\mathfrak{b}}_{i-1}$ for each $i = 1, \dots, r$. Let $a \in (\mathfrak{b}_{i-1} : f_i)$ with $\nu(a) = n$. Either $a^*f_i{}^* = 0$ or $a^*f_i{}^* = (af_i)^*$. In both cases, we have $a^*f_i^* \in (\mathfrak{b}_{i-1})^* = (f_1^*, \dots, f_{i-1}^*)$; hence

[1] Not to be confused with $\mathfrak{b}_{r-1}^*$, the set of homogeneous elements in $\mathfrak{b}^*$ of degree $r - 1$, defined in earlier section.

$\mathfrak{a}^* \in (\mathfrak{b}_{i-1})^*$. This implies $\mathfrak{a} \in \mathfrak{b}_{i-1} + \mathfrak{a}^{n+1} \cap (\mathfrak{b}_{i-1} : f_i)$ and hence we can write $a = x + a'$ for some $x \in \mathfrak{b}_{i-1}$ and $a' \in \mathfrak{a}^{n+1} \cap (\mathfrak{b}_{i-1} : f_i)$ with $\nu(a') = t \geqslant n+1$. Now repeating the argument, one can show $a' \in \mathfrak{b}_{i-1} + \mathfrak{a}^{t+1} \cap (\mathfrak{b}_{i-1} : f_i) \subseteq \mathfrak{b}_{i-1} + \mathfrak{a}^{n+2} \cap (\mathfrak{b}_{i-1} : f_i)$ and so on. Hence $a \in \bar{\mathfrak{b}}_{i-1}$.

$(ii) \Rightarrow (i)$: Conversely note that $p_i < \infty$ for each $i = 1, \dots, r$. Indeed, suppose on the contrary that $p_i = \infty$ for some $i \in \{1, \dots, r\}$. Then $f_i \in \bigcap_{n=1}^{\infty} \mathfrak{a}^n$ and by Krull's intersection theorem, there exists $a \in \mathfrak{a}$ such that $(1-a)f_i = 0$. Thus, $(1-a) \in (\mathfrak{b}_{i-1} : f_i) \subseteq \bar{\mathfrak{b}}_{i-1} = \bigcap_{n=1}^{\infty} \mathfrak{b}_{i-1} + \mathfrak{a}^n$. Now another application of Krull's intersection theorem over $A/\mathfrak{b}_i$ yields $a' \in \mathfrak{a}$ such that $(1-a')(1-a) \in \mathfrak{b}_{i-1}$. But this implies the existence of $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ such that $f + g = 1$, a contradiction since $\mathfrak{a} + \mathfrak{b} \neq A$.

Now let $a^* f_i^* \in (f_1^*, \dots, f_{i-1}^*) = (\mathfrak{b}_{i-1})^*$ with $\nu(a) = n$, then $af_i \in \mathfrak{b}_{i-1} + \mathfrak{a}^{n+p_i+1}$. Write $af_i = -\sum_{j=1}^{i-1} a_j f_j + b$ with $\sum_{j=1}^{i-1} a_j f_j \in \mathfrak{b}_{i-1}$ and $b \in \mathfrak{a}^{n+p_i+1}$. Then $b = af_i + \sum_{j=1}^{i-1} a_j f_j \in \mathfrak{b}_i \cap \mathfrak{a}^{n+p_i+1}$.

Since $b \in \mathfrak{b}_i$, we can write $b = \sum_{j=1}^{i} b_j f_j$ where each $b_j \in \mathfrak{a}^{n+p_i+1-p_j}$. This implies $(a - b_i)f_i = \sum_{j=1}^{i-1}(b_j - a_j)f_j \in \mathfrak{b}_{i-1}$, so $a - b_i \in (\mathfrak{b}_{i-1} : f_i)$ and hence $a \in (\mathfrak{b}_{i-1} : f_i) + \mathfrak{a}^{n+1} \subseteq \bar{\mathfrak{b}}_{i-1} + \mathfrak{a}^{n+1}$.

Hence $a \in \mathfrak{b}_{i-1} \cap \mathfrak{a}^n + \mathfrak{a}^{n+1}$, which implies $a^* \in (\mathfrak{b}_{i-1})^* = (f_1^*, \dots, f_{i-1}^*)$. ◻

Corollary 3.4. *Let A be a local ring and $I, \mathfrak{a}$ ideals of A, such that I^* is generated by a $G_A(\mathfrak{a})$-sequence. Then I is generated by an A-sequence.*

Proof. Let I^* be generated by a $G_A(\mathfrak{a})$-sequence: $g_1, \dots, g_r$, hence grade $(I^*) = r$. Given all minimal generating sets of I^* have the same cardinality, one can write $I^* = (f_1^*, \dots, f_r^*)$ with $f_i \in I$. Since I^* contains a $G_A(\mathfrak{a})$-sequence of length r and $G_A(\mathfrak{a})$ is graded-local since A is local, it follows that $f_1^*, \dots, f_r^*$ is a $G_A(\mathfrak{a})$-sequence (see [2], Cor 17.7). Now by Theorem 3.3 and the fact A is a local ring, if $af_i \in (f_1, \dots, f_{i-1}) = \mathfrak{b}_{i-1} \Rightarrow a \in \mathfrak{b}_{i-1} : f_i \subseteq \bar{\mathfrak{b}}_{i-1} = \mathfrak{b}_{i-1}$, hence $f_1, \dots, f_r$ is an A-sequence.

Finally, since $I^* = (f_1^*, \dots, f_r^*)$, we have

$$I \cap \mathfrak{a}^n \subseteq \sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i + \mathfrak{a}^{n+1} \subseteq (f_1, \dots, f_r) + \mathfrak{a}^{n+1}$$

for all $n \geqslant 0$. In particular, the $n = 0$ case shows us that $I \subseteq (f_1, \dots, f_r) + \mathfrak{a}$ and intersecting both sides with I gives $I \subseteq (f_1, \dots, f_r) + \mathfrak{a} \cap I \subseteq (f_1, \dots, f_r) + \mathfrak{a}^2$. Repeating the argument, we see that $I \subseteq \overline{(f_1, \dots, f_r)} = (f_1, \dots, f_r)$. But $(f_1, \dots, f_r) \subseteq I$, so I is generated by the A-sequence $f_1, \dots, f_r$. ◻

Remark 3.5. If $f_1^*, \dots, f_r^*$ is a $G_A(\mathfrak{a})$-sequence, it is not necessarily true that $f_1, \dots, f_r$ form an A-sequence unless $I = \bar{I}$ for every ideal I contained in $\mathfrak{b}$. For example, let $A = k[x, y, z] = k[X, Y, Z]/(XZ, X - XY)$, $\mathfrak{a} = (y)$, $f = yz$, then since y is a nonzerodivisor in A, we have: $G_A(\mathfrak{a}) = (A/\mathfrak{a})[T] = k[Z, T]$, which is a domain. Hence the initial form of f is a nonzerodivisor in $G_A(\mathfrak{a})$, however $xf = 0$.

Proposition 3.6. *Let $\mathfrak{a}$ and $\mathfrak{b} = (f_1, \dots, f_r)$ be two ideals of A such that $f_1, \dots, f_r$ is an A-sequence and $\mathfrak{a}^n \cap \mathfrak{b} = \sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i$ for all $n \geqslant 0$. Suppose either $\mathfrak{b} \subseteq \mathfrak{a}$ or A is local.*

Then $\mathfrak{a}^n \cap \mathfrak{b}_i = \sum_{j=1}^{i} \mathfrak{a}^{n-p_j} f_j$ for each $i = 1, \dots, r$ and for all $n \geqslant 0$; thus ${f_1}^*, \dots, {f_r}^*$ is a $G_A(\mathfrak{a})$-sequence.

The following proof is taken from [3].

Proof. It is enough to show that $\mathfrak{a}^n \cap \mathfrak{b}_{r-1} = \sum_{j=1}^{r-1} \mathfrak{a}^{n-p_j} f_j$ for all $n \geqslant 0$ and we will prove it using induction on n. The case $n = 0$ is trivial. For $n > 0$,

$$\begin{aligned}\mathfrak{a}^n \cap \mathfrak{b}_{r-1} &= (\mathfrak{a}^n \cap \mathfrak{b}) \cap \mathfrak{b}_{r-1} \\ &= \left(\sum_{i=1}^{r-1} \mathfrak{a}^{n-p_i} f_i + \mathfrak{a}^{n-p_r} f_r \right) \cap \mathfrak{b}_{r-1} \\ &= \sum_{i=1}^{r-1} \mathfrak{a}^{n-p_i} f_i + \mathfrak{a}^{n-p_r} f_r \cap \mathfrak{b}_{r-1}.\end{aligned}$$

Since f_r is a nonzerodivisor on $A/\mathfrak{b}_{r-1}$, we have $\mathfrak{a}^{n-p_r} f_r \cap \mathfrak{b}_{r-1} = f_r(\mathfrak{a}^{n-p_r} \cap \mathfrak{b}_{r-1})$. Thus,

$$\mathfrak{a}^n \cap \mathfrak{b}_{r-1} = \sum_{i=1}^{r-1} \mathfrak{a}^{n-p_i} f_i + f_r(\mathfrak{a}^{n-p_r} \cap \mathfrak{b}_{r-1}).$$

If $\mathfrak{b} \subseteq \mathfrak{a}$, then $p_r \geqslant 1$; if A is local and $p_r = 0$, then we have

$$\mathfrak{a}^n \cap \mathfrak{b}_{r-1} = \sum_{i=1}^{r-1} \mathfrak{a}^{n-p_i} f_i + f_r(\mathfrak{a}^n \cap \mathfrak{b}_{r-1}),$$

so $\mathfrak{a}^n \cap \mathfrak{b}_{r-1} = \sum_{i=1}^{r-1} \mathfrak{a}^{n-p_i} f_i$ by Nakayama's lemma. If $p_r \geqslant 1$, then $n - p_r \leqslant n-1$, so the inductive hypothesis implies that

$$\mathfrak{a}^n \cap \mathfrak{b}_{r-1} = \sum_{i=1}^{r-1} \mathfrak{a}^{n-p_i} f_i + f_r \left(\sum_{j=1}^{r-1} \mathfrak{a}^{n-p_r-p_j} f_j \right) \subseteq \sum_{i=1}^{r-1} \mathfrak{a}^{n-p_i} f_i.$$

This completes the proof. □

Corollary 3.7. *Let $\mathfrak{a}$ and $\mathfrak{b} = (f_1, \dots, f_r)$ be two ideals of a local ring A. Then ${f_1}^*, \dots, {f_r}^*$ is a $G_A(\mathfrak{a})$-sequence if and only if $f_1, \dots, f_r$ is an A-sequence and moreover $\mathfrak{a}^n \cap \mathfrak{b} = \sum_{i=1}^{r} \mathfrak{a}^{n-p_i} f_i$ for all $n \geqslant 0$.*

Remark 3.8. Let $(A, \mathfrak{m})$ be a local ring and $z \in \mathfrak{m} - \mathfrak{m}^2$. The above corollary says that z^* is a nonzerodivisor in $G_A(\mathfrak{m})$ if and only if z is a nonzerodivisor in A and $\mathfrak{m}^n \cap (z) = \mathfrak{m}^{n-1} z$ for all $n \geqslant 0$. This was first proved by Hironaka ([4], Proposition 6).

4. Applications

Proposition 4.1. *Let $\mathfrak{a}$ and $\mathfrak{b} = (f_1, \dots, f_r)$ be ideals of A such that $\mathfrak{b} \subseteq \mathfrak{a}$, $f_1, \dots, f_r$ is an A-sequence and $\mathfrak{a}\mathfrak{b} = \mathfrak{a}^2$. Then the initial forms of the f_is form a $G_A(\mathfrak{a})$-sequence.*

Proof. We will first prove that $p_i = 1$ for all $1 \leqslant i \leqslant r$. Assume on the contrary that $f_i \in \mathfrak{a}^2 = \mathfrak{a}\mathfrak{b}$. Then $f_i = \sum_{j=1}^{r} a_j f_j$ with $a_j \in \mathfrak{a}$, so $\sum_{j \neq i} a_j f_j + (a_i - 1) f_i = 0$. But we know that if a regular sequence $f_1, \dots, f_r$ satisfies a homogeneous polynomial $F(Y_1, \dots, Y_r)$, then F has coefficients in the ideal $(f_1, \dots, f_r)$. Applying this fact to our situation, we see that $a_i - 1 \in \mathfrak{b} \subseteq \mathfrak{a}$, which contradicts the hypothesis $\mathfrak{a} \neq A$. Thus $f_i \notin \mathfrak{a}^2$, and hence $p_i = 1$ for all i.

Now by Proposition 3.6, it is enough to prove $\mathfrak{a}^n \cap \mathfrak{b} = \mathfrak{a}^{n-1}\mathfrak{b}$ for all $n \geqslant 0$. This is trivial for $n = 0$ and $n = 1$. For $n \geqslant 2$, we have $\mathfrak{a}^n = \mathfrak{a}^{n-1}\mathfrak{b}$, hence

$$\mathfrak{a}^n \cap \mathfrak{b} = \mathfrak{a}^{n-1}\mathfrak{b} \cap \mathfrak{b} = \mathfrak{a}^{n-1}\mathfrak{b}.$$

This completes the proof. □

Here is an interesting application of the above proposition: let $(A, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension r, with embedding dimension m and multiplicity e. It is known that $m \leqslant e+r-1$ and the equality holds if and only if there exists an A-sequence $f_1, \dots, f_r$ in $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}(f_1, \dots, f_r)$ ([7], Theorem 1). In this situation, the above proposition says that ${f_1}^*, \dots, {f_r}^*$ is a $G_A(\mathfrak{m})$-sequence and that is enough to conclude $G_A(\mathfrak{m})$ is Cohen-Macaulay [5].

Remark 4.2. A new and simplified proof of ([9], Theorem 3.2) is obtained from the results of this paper:

Let A be a Cohen-Macaulay ring and let $a_1, \dots, a_s$ be a regular sequence, $I = (a_1, \dots, a_s), t$ an integer $\geqslant 1$. Then $G_A(\mathfrak{a})$ is Cohen-Macaulay if $\mathfrak{a} = I^t$.

Proof. Without loss of generality, we may assume that $(A, \mathfrak{m})$ is an r-dimensional local ring [9]. Extend $a_1, \dots, a_s$ to a maximal A-sequence $a_1, \dots, a_s, f_{s+1}, \dots, f_r$ in $\mathfrak{m}$. Let $J = (f_{s+1}, \dots, f_r)$ and for $1 \leqslant i \leqslant s$, let $f_i = a_i^t$ and $\mathfrak{b} = (f_1, \dots, f_r)$. Since $f_1, \dots, f_s$ is a regular sequence on A/J, we have $\mathfrak{a}^n \cap \mathfrak{b} \subseteq \mathfrak{a}^{n-1}(f_1, \dots, f_s) + J$ for all $n \geqslant 0$ ([9], Lemma 2.1). Since $f_{s+1}, \dots, f_r$ is a regular sequence modulo I, hence modulo $\mathfrak{a}^n$ for all $n \geqslant 1$, we have $\mathfrak{a}^n \cap J = \mathfrak{a}^n J$. It then follows that $\mathfrak{a}^n \cap \mathfrak{b} \subseteq \mathfrak{a}^n \cap (\mathfrak{a}^{n-1}(f_1, \dots, f_s) + J) = \mathfrak{a}^{n-1}(f_1, \dots, f_s) + \mathfrak{a}^n \cap J = \mathfrak{a}^{n-1}(f_1, \dots, f_s) + \mathfrak{a}^n J$. Hence by Proposition 3.6, $f_1^*, \dots, f_r^*$ is a $G_A(\mathfrak{a})$-sequence and by [5], given $\dim G_A(\mathfrak{a}) = r$, it is enough to prove that $G_A(\mathfrak{a})$ is Cohen-Macaulay. □

Example 4.3. The initial form of the same element with respect to two different ideals may or may not be a zerodivisor. For example, let $A = k[\![x, y, z]\!] = k[\![X, Y, Z]\!]/(XY - Z^2)$, $\mathfrak{a} = (x, y, z)$, and $I = (x, z)$. The initial form of x with respect to I is $x + I^2$ and since $(x+I^2)(y+I) = xy+I^2 = z^2+I^2 = 0$, $x+I^2$ is a zerodivisor. On the other hand, the initial form of x with respect to $\mathfrak{a}$ is a nonzerodivisor.

In the following proposition, let f^* denotes the initial form with respect to $\mathfrak{a}$ and f^0 the initial form with respect to I.

Proposition 4.4. *Let $I \subseteq \mathfrak{a}$ be ideals of A and let $f_1, \dots, f_r$ be elements of I such that $\nu_I(f_i) = \nu_{\mathfrak{a}}(f_i)$ for each i. Assume that $f_1^*, \dots, f_r^*$ form a $G_A(\mathfrak{a})$-sequence. Then $f_1^0, \dots, f_r^0$ form a minimal base of the ideal $(f_1^0, \dots, f_r^0)$ of $G_A(I)$.*

Proof. By ([1], Corollary 2.9), $f_1^*, \dots, f_r^*$ is a $G_A(\mathfrak{a})$-sequence up to permutation. Given $f_r^0 = \sum_{i=1}^{r-1} a_i^0 f_i^0$, let $a = \sum_{i=1}^{r-1} a_i f_i$ and $p = \nu_{\mathfrak{a}}(f_r) = \nu_I(f_r)$. Then we can write $f_r = a + b$, where $b \in I^{p+1}$. Therefore $a \in \mathfrak{a}^p$ and $a \notin \mathfrak{a}^{p+1}$; it follows that $f_r^* = a^* \in (f_1, \dots, f_{r-1})^* = (f_1^*, \dots, f_{r-1}^*)$, a contradiction. □

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